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Consistency and Convergence for Numerical Radiation Conditions

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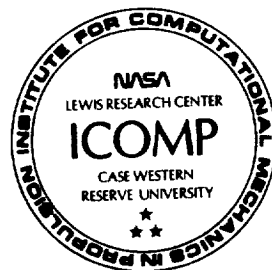
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CONSISTENCY AND CONVERGENCE FOR NUMERICAL RADIATION CONDITIONS *

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Abstract. We consider the problem of imposing radiation conditions at artificial boundaries for the numerical simulation of wave propagation. Our emphasis is on the behavior and analysis of the error which results from the restriction of the domain. We give a brief outline of the theory of error estimation for boundary conditions. Use is made of the asymptotic analysis of propagating wave groups to derive and analyze boundary operators. For dissipative problems this leads to local, accurate conditions, but falls short in the hyperbolic case. A numerical experiment on the solution of the wave equation with cylindrical symmetry is described. We give a unified presentation of a number of conditions which have been proposed in the literature and display the time dependence of the error which results from their use. The results are in qualitative agreement with theoretical considerations. We find, however, that for this model problem it is particularly difficult to force the error to decay rapidly in time.

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1. INTRODUCTION

1. Introduction. It is a common feature of many wave propagation problems that they are naturally posed on unbounded spatial domains and that disturbances are radiated to the far field. The numerical solution of such problems requires their restriction to a finite computational domain, usually via the introduction of an artificial boundary. The choice of radiation boundary conditions there can have a dominant influence on the solution, particularly for long time simulations. For this reason, the study of numerical radiation conditions is an important aspect of the numerical analysis of wave propagation, and has been the subject of a great deal of research in recent years. In this paper, we discuss some aspects of the problem with an emphasis on the behavior of the error which results from introducing the artificial boundary and on the use of asymptotic analysis in its estimation.

The outline of a general theory of artificial boundary conditions is fairly straightforward to develop. Consider a linear problem on a cylindrical domain:

$$(1) \quad L(\partial_t, \partial_x, x)u = f, \quad x = (x_1, x_2, \dots) \in [0, \infty) \times \Omega, \quad t \geq 0,$$

supplemented by appropriate boundary and initial conditions. (Note that coordinate systems can be chosen so that the far field of exterior domains is transformed into a cylinder.) Suppose an artificial boundary is introduced at $x_1 = \tau$ and the radiation condition:

$$(2) \quad Bv = g,$$

is imposed on the approximate, finite domain solution v . The error resulting from the restriction of the domain, $e = u - v$, satisfies:

$$(3) \quad Le = 0, \quad x \in [0, \tau] \times \Omega, \quad t \geq 0,$$

$$(4) \quad Be = Bu - g, \quad x_1 = \tau,$$

in addition to homogeneous initial conditions and homogeneous conditions at $\partial\Omega$ and at $x_1 = 0$. To estimate $\|e\|$ we need appropriate generalizations of the usual concepts from numerical analysis, stability and consistency. Stability in this case is simply the well-posedness of (3-4), which we take to be the existence of a finite K independent of the data $Bu - g$ such that,

$$(5) \quad \|e\| \leq K\|Bu - g\|.$$

Here the particular norms are determined by the differential operators L and B . In order to take account of any decay of the error off the artificial boundary, use of weighted norms may be of interest. Stability can be established, in principle, by studying Kreiss type conditions. Given stability, we see that the error will be small if $\|Bu - g\|$ is; that is if the solution of the original problem approximately satisfies the radiation condition. Carrying further the analogy with the theory of discretization error, we may consider a sequence of approximate problems depending on a parameter $\epsilon \rightarrow 0$ such that,

$$(6) \quad \|B(\epsilon)u - g(\epsilon)\| = o((K(\epsilon))^{-1}),$$

which is simply a consistency condition. Clearly, (5) and (6) together imply the convergence of $\|e\|$ to 0.

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The problem of designing effective approximate radiation conditions is, then, to find B and g such that the stability condition is satisfied and $\|Bu - g\|$ is small. There is the additional consideration that the finite domain problem for v be reasonably easy to solve, which rules out arbitrary nonlocality in space and time for the operator B . The error analysis outlined above, though not typically discussed in papers on numerical radiation conditions, has been carried out in a few cases. Halpern and Rauch [11] consider general conditions for the wave equation which are exact in the high frequency limit and show that the error is small if the average frequency present is high. Bayliss and Turkel [2] use a long range asymptotic expansion of solutions of the wave equation to derive error estimates in powers of τ^{-1} . In joint work with Hariharan [10] we generalize these to convective wave equations. In all the cases above, as will be seen in the numerical example later on, the error estimates are nonuniform in time. Error estimates in τ^{-1} are also given in [9] for dissipative problems. These will be discussed in more detail in the next section.

2. Asymptotic Analysis and Group Velocity. To make further progress we suppose that the coefficients of L are independent of x_1 and t for $x_1 \geq \tau$. Then, using Laplace transforms and eigenfunction expansions, u may be represented as a superposition of normal modes,

$$(7) \quad u = \sum_l \int_0^t c_l(p) q_l(t - p, x) dp, \quad x_1 \geq \tau$$

$$(8) \quad q_l(t, x) = \frac{1}{2\pi i} \int_C e^{st + \lambda_l(s)x_1} y_l(x_2, \dots; s) ds.$$

(For further details of this representation and the following arguments see [9].) If L is second order in x_1 , an exact boundary condition satisfied by u is given by:

$$(9) \quad \frac{d\hat{u}_l}{dx_1} = \lambda_l(s) \hat{u}_l.$$

The generalized dispersion relation, $\lambda_l(s)$, which defines the exact boundary operator for the l th mode, is difficult to obtain in analytic form except for problems with constant coefficients. Furthermore, (9) defines an operator which is nonlocal in both space and time and, hence, which violates the condition that the reduced domain problem be easily solved. To get an operator which is local in time, we must approximate $\lambda_l(s)$ by a rational function in s . (Typically, locality in space is also obtained by restriction to a small number of modes or approximating the l dependence.) Let the approximate condition be expressed by:

$$(10) \quad \frac{d\hat{v}_l}{dx_1} = b_l(s) \hat{v}_l.$$

Then the transform of Bu , which must be made small to get a good error estimate, is given by:

$$(11) \quad \widehat{Bu}_l = (\lambda_l(s) - b_l(s)) \hat{u}_l.$$

One way to make (11) small would be to make $\lambda_l - b_l$ small. However, we cannot expect to be able to enforce this everywhere in s space, as a low degree rational approximation

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to an arbitrary function is, in general, only accurate in restricted regions. We must, then, choose regions in s space where λ_l is to be approximated. Clearly, these should correspond to regions where \hat{u}_l is large. It is in identifying these that asymptotic analysis can play an important role. We note that the complementary problem of determining the stability of rational approximations is taken up in detail for a particular case by Trefethen and Halpern [12].

Asymptotic expansions of u_l for τ large can be computed by the method of steepest descent and lead to a study of wave groups. These are defined by:

$$(12) \quad \lambda'_l(s) = -\gamma,$$

where γ is real and nonnegative. We suppose that (12) defines curves, $s = s^*(\gamma)$. To derive boundary conditions, we must further restrict attention to small subintervals of these curves. We note that each wave group has an exponential decay rate, $\mu_l(\gamma)$, given by:

$$(13) \quad \mu_l(\gamma) = -\Re(s^*(\gamma)\gamma + \lambda_l(s^*(\gamma))).$$

We expect the large x_1 behavior of the solution to be dominated by wave groups with minimum decay rates, if they exist. Differentiating μ_l with respect to γ yields:

$$(14) \quad \mu'_l(\gamma) = -\Re(s^*).$$

This leads to the consideration of two typical cases:

Case 1: The curve $s^*(\gamma)$ coincides with part of the imaginary s axis.

From (14) we see that the decay rates are independent of the group velocity. This indicates that all wave groups may be important in the far field and, therefore, provides no guidance to the local approximation of the dispersion relation. Case 1 holds for hyperbolic problems. Consider, for example, the dispersive wave equation,

$$(15) \quad u_{tt} - u_{xx} + u = 0,$$

which has the dispersion relation:

$$(16) \quad \lambda = -(1 + s^2)^{\frac{1}{2}}.$$

Solving (12) we find that curves of real, positive group velocity are given by the portions of the imaginary axis above i and below $-i$, where the decay rates are $\mu = 0$, and the nonnegative real axis where μ is positive. It is worth noting that the question of finding accurate and efficient boundary conditions for this simple problem has never been satisfactorily settled. It has recently been taken up again by Engquist and Halpern [4] who propose an approximation to λ which is accurate for s large and at $s = 0$.

Case 2: The curve(s) $s^*(\gamma)$ cross the imaginary s axis.

In this case (14) implies a local minimum (or maximum) of the decay rate. An accurate asymptotic boundary operator can then be constructed by approximating the dispersion

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relation near the crossing point. In [9] we consider a linear approximation which leads to particularly simple boundary operators involving first order derivatives in space and time. Assuming a global minimum for the decay rate, we were able to show that the error is proportional to τ^{-1} . Applications of this technique to simulations of diffusion and advection by a parallel flow [7] and the propagation of disturbances to Poiseuille flow [8] have illustrated its effectiveness. Case 2 typically holds for problems with dissipation. The simplest example is the advection-diffusion equation in one dimension:

$$(17) \quad u_t + u_x - u_{xx} = 0,$$

with dispersion relation:

$$(18) \quad \lambda = \frac{1}{2} \left(1 - (1 + 4s)^{\frac{1}{2}} \right).$$

The curve of real group velocity coincides with the real s axis to the right of $s = -\frac{1}{4}$, so that the decay rate has a minimum at $s = 0$. A boundary condition, b , which agrees with λ in a neighborhood of 0 will be accurate.

It is of interest to examine the connection between the two cases by considering a dissipative problem which approaches the dispersive wave equation as a dissipation parameter approaches zero. We take, as a particular example,

$$(19) \quad \left(\frac{\partial}{\partial t} - \nu \left(\frac{\partial^2}{\partial x^2} - \alpha^2 \right) \right)^2 u - \frac{\partial^2 u}{\partial x^2} + \alpha^2 u = 0.$$

The dispersion relation has two branches:

$$(20) \quad \lambda_{\pm} = -\frac{1}{\sqrt{2\nu}} \left(1 + 2\nu s + 2\nu^2 \alpha^2 \pm (1 + 4\nu s)^{\frac{1}{2}} \right)^{\frac{1}{2}}.$$

For ν small λ_+ is large and is associated with rapidly decaying wave groups. It is wave groups defined by λ_- which are most important. We find two curves of real group velocity. The first coincides with the real s axis to the right of $-\frac{1}{4\nu}$. The minimal decay rate, $\lambda_-(0)$, is $-\alpha$. A second curve can be found numerically. Its ends are the branch points of λ_- , $s_{\pm} = -\nu\alpha^2 \pm i\alpha$. Following the curve from s_+ , the real and imaginary parts of s increase as we cross the imaginary axis. Eventually, the imaginary part reaches a maximum and decreases until the curve crosses the real axis. Expanding λ_- for ν small we find that the crossings take place at $s = \pm\sqrt{\frac{3}{2}}i\alpha + O(\nu)$. A local approximation to λ_+ is given by:

$$(21) \quad \lambda_- \sim - \left(\pm \frac{i\alpha}{\sqrt{2}} + \frac{\sqrt{27}}{2} \nu \alpha^2 + \sqrt{3}(s \mp \sqrt{\frac{3}{2}}i\alpha) \right).$$

We see that the decay rates are $O(\nu)$ so that these waves decay far less rapidly than the branch through $s = 0$. To derive boundary conditions which match the behavior of λ_- at both crossings we may either consider products of first order conditions [9] or direct rational approximations. Following the latter approach we find:

$$(22) \quad b = -\frac{4s^2 + 9\nu\alpha^2 s + 3\alpha^2}{\sqrt{12}s}.$$

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In practice, modifications of this condition to remove the singularity at 0 may be necessary. The corresponding differential operator is:

$$(23) \quad B = \left(\sqrt{12} \frac{\partial^2}{\partial x \partial t} + 4 \frac{\partial^2}{\partial t^2} + 9\nu\alpha^2 \frac{\partial}{\partial t} + 3\alpha^2 \right).$$

Of course we require two conditions for a fourth order equation, but the additional condition is easily found. (For example, by differentiating the first.) It is tempting to use the $\nu = 0$ limit of (22) for the dispersive wave equation. However it must be noted that the asymptotic error estimates are not uniformly valid as $\nu \rightarrow 0$. The curves of real group velocity steepen and approach tangency to the imaginary axis. This implies that the decay rates are nearly constant. Nonetheless, it would be interesting to experiment with these conditions for ν small, and we plan to do so in the future.

3. The Cylindrical Wave Equation: A Case Study. As we have seen, the general asymptotic analysis of propagating wave groups described above does not yield a technique for computing accurate, local radiation boundary conditions in the hyperbolic case. However a number of results have appeared in the literature related to the accuracy of radiation conditions for the wave equation. As mentioned earlier, the work of Bayliss and Turkel [2] for exterior domains contains error estimates in τ^{-1} . These are nonuniform in time. More recently, Engquist and Halpern [4],[5], Barry, Bielak and MacCamy [1] and Bielak and MacCamy [3] have considered the long time behavior and introduced modifications of conditions based on high frequency asymptotics to ensure the decay or at least suppress the growth of the error.

To better understand both the long and short time behavior of the error for various boundary conditions, we have considered the following simple test problem:

$$(24) \quad \frac{\partial^2 u}{\partial t^2} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right), \quad r \geq 1, \quad t \geq 0,$$

$$(25) \quad u(1, t) = 1 - \frac{\cos 2\pi t}{1 + t^2}, \quad u(r, 0) = \frac{\partial u}{\partial t}(r, 0) = 0.$$

An exact condition satisfied by u is:

$$(26) \quad \frac{d\hat{u}}{dr} = \frac{sK'_0(\tau s)}{K_0(\tau s)} \hat{u} \equiv \frac{1}{\tau} \mathcal{F}(z) \hat{u}.$$

Here $z = \tau s$. All boundary conditions we consider may be written as approximations, $b(z)$, to \mathcal{F} . For τ large we must consider the behavior of \mathcal{F} for z large:

$$(27) \quad \mathcal{F} = -z - \frac{1}{2} + \dots, \quad z \rightarrow \infty.$$

The long time behavior, however, corresponds to the limit $z \rightarrow 0$ where we have:

$$(28) \quad \mathcal{F} = \frac{1}{\ln z} + \dots, \quad z \rightarrow 0.$$

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We now list the conditions we have used in the experiment and compute their corresponding b .

Condition 1:

$$(29) \quad \frac{\partial v}{\partial r} + \frac{\partial v}{\partial t} = 0.$$

This simple condition follows, for example, from adding a correction to condition (32) to make $b(0) = 0$ as suggested in [4]. The properties of b are:

$$(30) \quad b(z) = -z,$$

$$(31) \quad b - \mathcal{F} = \begin{cases} \frac{1}{2}, & z \rightarrow \infty \\ O(\frac{1}{\ln z}), & z \rightarrow 0 \end{cases}.$$

Condition 2:

$$(32) \quad \frac{\partial v}{\partial r} + \frac{\partial v}{\partial t} + \frac{v}{2r} = 0.$$

This condition follows from the high frequency analysis of Engquist and Majda [6] and is the first in the hierarchy of conditions proposed by Bayliss and Turkel [2]. The properties of its b are:

$$(33) \quad b(z) = -z - \frac{1}{2},$$

$$(34) \quad b - \mathcal{F} = \begin{cases} O(z^{-1}), & z \rightarrow \infty \\ -\frac{1}{2}, & z \rightarrow 0 \end{cases}.$$

Condition 3:

$$(35) \quad \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial t} + \frac{5}{2r} \right) \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial t} + \frac{1}{2r} \right) v = 0.$$

This is the second condition in the Bayliss-Turkel hierarchy. It is chosen to match more terms in the large z expansion of \mathcal{F} . The results of [2] imply, for fixed time, an error proportional to τ^{-1} for Condition 2 and proportional to τ^{-2} for Condition 3. In order to compute b the differential equation is used to replace the derivatives of second order in r . This yields:

$$(36) \quad b(z) = -\frac{(z^2 + \frac{3z}{2} + \frac{3}{8})}{(z+1)},$$

$$(37) \quad b - \mathcal{F} = \begin{cases} O(z^{-3}), & z \rightarrow \infty \\ -\frac{3}{8}, & z \rightarrow 0 \end{cases}.$$

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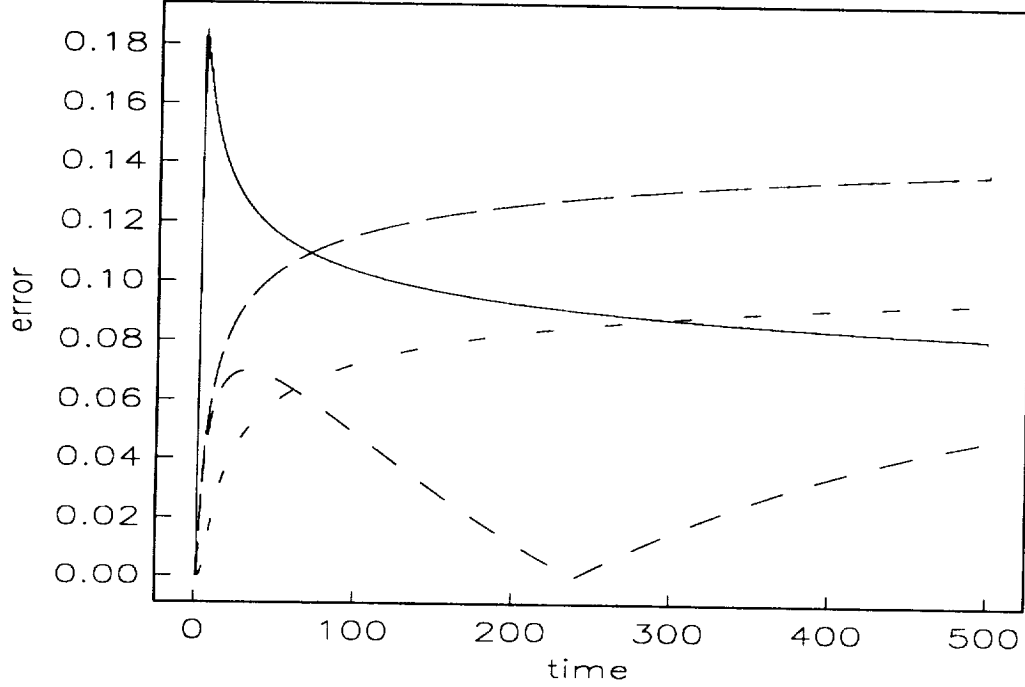


FIG. 1. Errors for $\tau = 2$. Condition 1: Solid line, Condition 2: Long dash - short space, Condition 3: Short dash - long space, Condition 4: Equal dash and space.

Condition 4:

$$(38) \quad \left(\frac{\partial}{\partial t} + \delta \right) \frac{\partial v}{\partial r} + \left(\frac{\partial}{\partial t} + \frac{1}{2\tau} + \delta \right) \frac{\partial v}{\partial t} = 0.$$

This condition is a generalization of those proposed by Barry, Bielak and MacCamy [1]. Here $\delta > 0$ is a free parameter. The results are somewhat sensitive to its choice. We didn't make a concerted effort to find an optimal δ , but do present the best results we obtained. These were for $\delta = .005$. Note that this condition captures the first two terms of the large z expansion and agrees with \mathcal{F} at $z = 0$. We have:

$$(39) \quad b(z) = -\frac{z(z + \frac{1}{2} + \delta\tau)}{(z + \delta\tau)},$$

$$(40) \quad b - \mathcal{F} = \begin{cases} O(z^{-1}), & z \rightarrow \infty \\ O(\frac{1}{\ln z}), & z \rightarrow 0 \end{cases}.$$

For the numerical experiments we used second order finite difference discretizations with a time step of .01 and a mesh width of .02. The artificial boundary was placed at $\tau = 2, 4, 8$. The errors were measured by comparison with a solution for $\tau = 252$ at $r = 1.3$ and $r = 1.8$.

The results are summarized in Figures 1-3. The maximum error for Condition 1 is largest, peaking at early times and slowly decaying for large time. Condition 2 is better for short times but has an error which grows monotonically. Condition 3 is by far the

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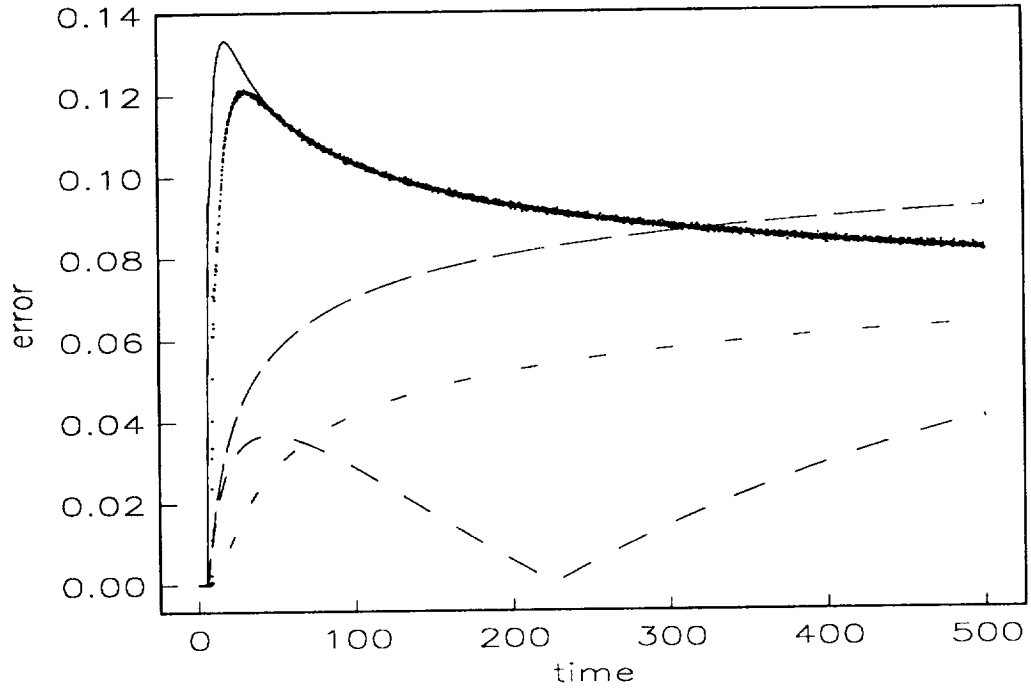


FIG. 2. Errors for $\tau = 4$. Condition 1: Solid line, Condition 2: Long dash - short space, Condition 3: Short dash - long space, Condition 4: Equal dash and space. The plotted points correspond to Condition 1 with a mesh stretched by a factor of 1.01 beyond $r = 2$.

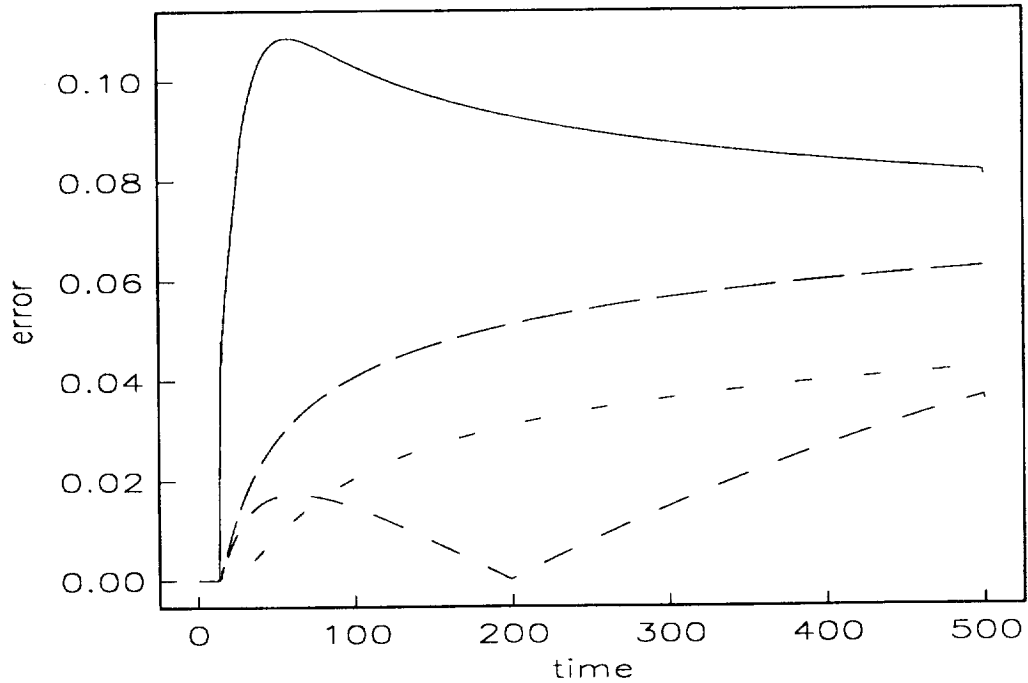


FIG. 3. Errors for $\tau = 8$. Condition 1: Solid line, Condition 2: Long dash - short space, Condition 3: Short dash - long space, Condition 4: Equal dash and space.

REFERENCES

best for short times but with a monotonically growing error also. Condition 4 exhibits the best overall performance. Its error displays oscillatory behavior. We also carried out a computation with mesh stretching, the results being displayed in Figure 3. We found that the largest mesh width had to be kept fairly small to get reasonable results. Note that the long time errors decay slowly as τ increases. The decay for Conditions 2 and 3 predicted by the analysis of [2] is, in particular, not evident. This is a result of the long time nonuniformity of the high frequency - large τ asymptotics. Qualitatively, these results are predicted by the analysis of the behavior of $b - \mathcal{F}$ for large and small z .

In no case do we observe a very rapid decay of the error with time. This is explained by the behavior of $b - \mathcal{F}$ for z small. As the $\frac{1}{\ln z}$ term cannot be canceled by a rational function, $b, \frac{d}{dz}(b - \mathcal{F})$ will always be singular at the origin. This implies a lower bound on the decay in time of the inverse transform of $(b - \mathcal{F})$. The only way around this difficulty is to use an operator which is nonlocal in time. In joint work with Hariharan and MacCamy, the derivation, analysis and efficient implementation of such a boundary operator is currently under consideration.

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